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# ON SOME COLOR PARTITION IDENTITIES OF SCHUR-TYPE OVERPARTITIONS

**K. C. Ajeyakashi, C. Shivashankar* and S. Chandankumar**

Department of Mathematics and Statistics,
M. S. Ramaiah University of Applied Sciences,
Peenya Campus, Bengaluru - 560058, Karnataka, INDIA

E-mail : 19MPRP727001@msruas.ac.in, chandan.s17@gmail.com

*Department of Mathematics,
Vidyavardhaka College of Engineering,
Gokulam III Stage, Mysuru, Karnataka - 570002, INDIA

E-mail : shankars224@gmail.com

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**Abstract:** Motivated by the work of Broudy and Lovejoy, who investigated arithmetic properties of an overpartition counting function $S(n)$ that arose in connection with Schur's partition theorem and a universal mock theta function. In this article, we present a few Ramanujan-type color partition identities of $S(n)$. Our proof relies on basic identities of theta-functions.

**Keywords and Phrases:** Overpartitions, Schur-type overpartitions, Color partitions.

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## 1. Introduction

The celebrated Rogers-Ramanujan identities, which show the equality of a hypergeometric $q-$ series and an infinite product, for $i = 1$ or 2 is stated as

$$\sum_{n\geq 0} L_i(n)q^n = \frac{1}{(q^i;q^5)_\infty (q^{5-i};q^5)_\infty},$$

where $L_i(n)$ denotes the number of partitions of $n$ where parts are at least $i$ and differ by at least 2. As usual for any complex numbers $a$ and $q$ with $|q| < 1$, we define the infinite $q-$product as

$$(a;q)_\infty := \prod_{k=0}^{\infty} \left(1 - aq^k\right).$$

Schur [11] proved a similar identity for partitions with parts differing by at least 3. The Schur's result was extended by Gleissberg [9] to a general modulus, the generating function is denoted by

$$\mathscr{B}_{d,r}(q) = \sum_{n\geq 0} B_{d,r}(n)q^n,$$

where $B_{d,r}(n)$ denote the number of partitions of $n$ such that each part is congruent to $0, \pm r \pmod d$, each pair of parts differ by at least $d$, and if $d \mid \lambda_i$, then $\lambda_i - \lambda_{i+1} > d$. Further, let $\mathscr{E}_{d,r}(q)$ denote the generating function for partitions into distinct parts that are congruent to $\pm r \pmod d$, with enumeration function $E_{d,r}(n)$, given by

$$\mathscr{E}_{d,r}(q) = \sum_{n\geq 0} E_{d,r}(n)q^n = (-q^r;q^d)_\infty(-q^{d-r};q^d)_\infty.$$

Following Schur, we define his general identity which is stated in the following theorem

**Theorem 1.1.** *For $d \geq 3, 1 \leq r < \frac{d}{2}$, then*

$$\mathscr{B}_{d,r}(q) = \mathscr{E}_{d,r}(q).$$

It is worth a while to have a look at an interesting article by K. Bringmann and K. Mahlburg [5], where the authors have studied the combinatorics of the Schur-type enumeration functions and derived $q-$ difference equations whose solution gave rise to $q-$series expression for the generating function $\mathscr{B}_{d,r}(q)$. Motivated by this work in [4], Bringmann, J. Lovejoy and Malburg have proved analytic and combinatorial identities which are reminiscent of Schur's classical partition theorem. Specially, they have shown that certain families of overpartitons whose parts satisfy gap conditions are equinumerous with partitions whose parts satisfy certain congruence conditions.

In the theory of partitions, there are various results such as Schur's theorem and the Rogers–Ramanujan identities. Naturally, studying the arithmetic properties of the associated counting functions is of great interest. One such investigation was carried out by Broudy and Lovejoy [6]. They studied arithmetic properties of a certain counting function $S(n)$, which first arose in an overpartition identity connected with Schur's partition theorem and a universal mock theta function. In particular, they proved a number of arithmetic properties satisfied by $S(n)$ modulo 2, 4, and 5 in various arithmetic progressions.

Recently, Chern, da Silva, and Sellers [7] extended the work of Broudy and Lovejoy on the Schur-type overpartition function $S(n)$. Using only elementary $q$-series identities, generating-function dissections, and the Alaca–Alaca–Williams parameterization, they established new arithmetic properties of $S(n)$ modulo small powers of 2. They also obtained internal congruences modulo 8 and presented conjectural congruences modulo 32 supported by computational evidence.

By a partition $\lambda$ of $n$ we mean a non-decreasing sequence of integer parts $1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ that sum to $n$; see [1] for further background.

Recall that an overpartition $\lambda$ of $n$ is a partition of $n$ in which the final occurrence of an integer may be overlined. Define the $4\times 4$ matrix $\overline{A}_{d,r}$ by

$$\overline{A}_{d,r} = \begin{array}{c} \\ \scriptstyle\overline{r} \\ \scriptstyle\overline{d-r} \\ \scriptstyle\overline{d} \\ \scriptstyle d \end{array} \begin{array}{c} \begin{array}{cccc} \scriptstyle\overline{r} & \scriptstyle\overline{d-r} & \scriptstyle\overline{d} & \scriptstyle d \end{array} \\ \begin{pmatrix} d & 2r & d+r & r \\ 2d-2r & d & 2d-r & d-r \\ 2d-r & d+r & 2d & d \\ d-r & r & d & 0 \end{pmatrix} \end{array}$$

We consider overpartitions into parts congruent to $r, d-r$, or $d \pmod{d}$, where only multiples of $d$ may appear non-overlined. For $n \geq 0$, let $\overline{B}_{d,r}(n)$ denote the number of such overpartitions $\lambda$ of $n$ where

(i) The smallest part is $\overline{r}, \overline{d-r}, \overline{d}$, or $2d$ modulo $2d$;

(ii) For $u, v \in \{\overline{r}, \overline{d-r}, \overline{d}, d\}$ if $\lambda_{i+1} \equiv u \pmod{d}$ and $\lambda_i \equiv v \pmod{d}$, then $\lambda_{i+1} - \lambda_i \geq \overline{A}_{d,r}(u,v)$;

(iii) For $u, v \in \{\overline{r}, \overline{d-r}, \overline{d}, d\}$ if $\lambda_{i+1} \equiv u \pmod{d}$ and $\lambda_i \equiv v \pmod{d}$, then $\lambda_{i+1} - \lambda_i \equiv \overline{A}_{d,r}(u,v) \pmod{2d}$. In words, the actual difference between two parts must be congruent modulo $2d$ to the smallest allowable difference.

The generating function for $\overline{B}_{d,r}(n)$ is defined by

$$\sum_{n\geq 0} \overline{B}_{d,r}(n)q^n = \frac{(-q^r;q^d)_\infty(-q^{d-r};q^d)_\infty}{(q^{2d};q^{2d})_\infty}. \tag{1.1}$$

A special case of the above identity with $d = 3, r = 1$ is written as

$$\sum_{n\geq 0} S(n)q^n = \frac{1}{(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty}, \tag{1.2}$$

where $S(n)$ denote the number of overpartitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of $n$ where only parts divisible by 3 may occur non-overlined with the conditions:

1. The smallest part is $\overline{1}, \overline{2}, \overline{3}$, or $6 \pmod{6}$;

2. For $u, v \in \{\overline{1}, \overline{2}, \overline{3}, 3\}$, if $\lambda_i \equiv u \pmod{3}$ and $\lambda_{i+1} \equiv v \pmod{3}$, then
$$\lambda_i - \lambda_{i+1} \geq A_{3,1}(u,v);$$

3. For $u, v \in \{\overline{1}, \overline{2}, \overline{3}, 3\}$, if $\lambda_i \equiv u \pmod{3}$ and $\lambda_{i+1} \equiv v \pmod{3}$, then
$$\lambda_i - \lambda_{i+1} \equiv A_{3,1}(u,v) \pmod{6}.$$

   In words, the *actual difference* between two parts must be congruent modulo 6 to the *smallest allowable difference.*

Motivated by these works, we present several color partition identities for $S(n)$.

Throughout this paper, we will use the following standard $q-$series notation and $f_k$ is defined as

$$f_k := \left(q^k, q^k\right)_\infty .$$

Following Ramanujan, we define his general theta–function $f(a,b)$ as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ \ |ab|<1.$$

Jacobi's triple product identity takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Two special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

In view of the equation (1.2), the generating function for $S(n)$ can be expressed as

$$\sum_{n=0}^{\infty} S(n)q^n = \frac{f_2 f_3}{f_1 f_6^2}. \tag{1.3}$$

In this article, we study a few color partition identities of Schur-type overpartitions $S(n)$. We have proved some Ramanujan-like congruences. The generating functions for two colored and three colored partition of $S(n)$ are respectively denoted as $S_2(n)$ and $S_3(n)$ and defined as

$$\sum_{n=0}^{\infty} S_2(n)q^n = \frac{f_2^2 f_3^2}{f_1^2 f_6^4}, \tag{1.4}$$

$$\sum_{n=0}^{\infty} S_3(n)q^n = \frac{f_2^3 f_3^3}{f_1^3 f_6^6}. \tag{1.5}$$

This paper is structured as follows. In section 2, we discuss a few dissection formulae that are important to establish our main results. In section 3, we derive Ramanujan-like congruences for modulo 4, 20 and some infinite families of congruences modulo 8. In section 4, we derive Ramanujan-like congruences for modulo 3, 4 and 9 including some infinite families of congruences modulo 9.

## 2. Preliminary results

We need the following lemmas and definitions to prove our main results. The binomial theorem, for each positive integer $k$ and $m$,

$$f_k^{2m} \equiv f_{2k}^m \pmod{2^m}, \tag{2.1}$$

$$f_k^{3m} \equiv f_{3k}^m \pmod{3^m}, \tag{2.2}$$

$$f_k^{5m} \equiv f_{5k}^m \pmod{5^m}. \tag{2.3}$$

**Lemma 2.1.** *The following 2-dissections hold*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.4}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.5}$$

Lemma (2.1) is a consequence of dissection formulas of Ramanujan, collected from Berndt's book [2. p. 40, Entry 25].

**Lemma 2.2.** *The following 2-dissections hold*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{2.6}$$

$$\frac{1}{f_1^2 f_3^2} = \frac{f_8^5 f_{24}^5}{f_2^5 f_6^5 f_{16}^2 f_{48}^2} + 2q\frac{f_4^4 f_{12}^4}{f_2^6 f_6^6} + 4q^4\frac{f_4^2 f_{12}^2 f_{16}^4 f_{48}^2}{f_2^5 f_6^5 f_8 f_{24}}, \tag{2.7}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{2.8}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{2.9}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}. \tag{2.10}$$

For the proof of the above Lemma 2.2, one can refer [3].

**Lemma 2.3.** *The following* 3*-dissection hold*

$$\frac{f_2^3}{f_1^3} = \frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}. \tag{2.11}$$

Equation (2.11) is collected from [12].

**Lemma 2.4.** [2, p. 49] *The following* 3*-dissection hold*

$$\varphi(q) = \varphi(q^9) + 2qf\left(q^3, q^{15}\right) = \frac{f_{18}^5}{f_9^2 f_{36}^2} + 2q\frac{f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}}, \tag{2.12}$$

$$\varphi(-q) = \varphi(-q^9) - 2qf\left(-q^3, -q^{15}\right). \tag{2.13}$$

**Lemma 2.5.** ([8, Theorem 2.2]) *For any prime* $p \geq 5$*,*

$$f_1 = \sum_{\substack{k=\frac{1-p}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2},$$

*where*

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases} \tag{2.14}$$

**Lemma 2.6.** ([8, Theorem 2.1]) *For any odd prime* $p$*,*

$$\frac{f_2^2}{f_1} = \sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \frac{f_{2p^2}^2}{f_{p^2}}. \tag{2.15}$$

Furthermore,

$$\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod p \text{ for } 0 \leq m \leq \frac{p-3}{2}.$$

**Lemma 2.7.** ([13, Theorem 2.1] *For any odd prime* $p$

$$f_1^3 = \sum_{k=-\frac{p-1}{2}}^{\frac{p-3}{2}} (-1)^k q^{\frac{k^2+k}{2}} B_k(q^p) + (-1)^{\frac{p-1}{2}} p q^{\frac{p^2-1}{8}} f_{p^2}^3, \tag{2.16}$$

where

$$B_k(q) := \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n(2pn+2k+1)q^{\dfrac{pn^2+(2k+1)n}{2}}.$$

Furthermore,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p} \text{ for } -\frac{p-1}{2} \le k \le \frac{p-3}{2}.$$

We establish our results in the parts that follow using the preliminary results.

## 3. Two color partition identities of $S(n)$

**Theorem 3.1.** *We have*

$$\sum_{n=0}^{\infty} S_2\left(2n\right)q^n = \frac{f_2^4 f_6^2}{f_1^3 f_3^3 f_4 f_{12}}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} S_2\left(2n+1\right)q^n = 2\frac{f_2 f_4 f_{12}}{f_1^2 f_3^2 f_6}. \tag{3.2}$$

**Proof.** Substituting (2.8) into (1.4), we get

$$\sum_{n=0}^{\infty} S_2\left(n\right)q^n = \frac{f_2^2}{f_6^4}\left(\frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}\right) = \frac{f_4^4 f_{12}^2}{f_2^3 f_6^3 f_8 f_{24}} + 2q\frac{f_4 f_8 f_{24}}{f_2^2 f_6^2 f_{12}}. \tag{3.3}$$

Equations (3.1) and (3.2) follows by extracting odd and even powers of $q$ on both side of the above equation.

**Theorem 3.2.** *For any non–negative integer $n$, we have*

$$S_2(4n+3) \equiv 0 \pmod{4}. \tag{3.4}$$

**Proof.** Substituting (2.7) into (3.2), we obtain

$$\sum_{n=0}^{\infty} S_2\left(2n+1\right)q^n = 2\frac{f_{12}^{11} f_8^4}{f_2^3 f_6^9 f_4 f_{24}^4} + 4q\frac{f_4^5 f_{12}^5}{f_2^5 f_6^7} + 2q^2\frac{f_4^{11} f_{24}^4}{f_2^7 f_6^5 f_8^4 f_{12}}. \tag{3.5}$$

Extracting those terms containing the power $q^{2n}$ and $q^{2n+1}$ from both sides of (3.5) and then changing $q$ to $q^{\frac{1}{2}}$, we obtain

$$\sum_{n=0}^{\infty} S_2\left(4n+1\right)q^n = 2\frac{f_6^{11} f_4^4}{f_1^3 f_3^9 f_2 f_{12}^4} + 2q\frac{f_2^{11} f_{12}^4}{f_1^7 f_3^5 f_4^4 f_6} \tag{3.6}$$

and

$$\sum_{n=0}^{\infty} S_2\left(4n+3\right)q^n = 4\frac{f_2^5 f_6^5}{f_1^5 f_3^7}. \tag{3.7}$$

Congruence (3.4) follows from (3.7).

**Theorem 3.3.** *For any non–negative integer* $n$ *, we have*

$$S_2\left(20n+4s+3\right) \equiv 0 \pmod{20}, \tag{3.8}$$

*where* $s \in \{2,3,4\}$.

**Proof.** Applying (2.3) into (3.7), we get

$$\sum_{n=0}^{\infty} S_2\left(4n+3\right)q^n \equiv 4\frac{f_{10}f_{30}}{f_5 f_{15}}\frac{1}{f_3^2} \pmod{20}. \tag{3.9}$$

Let $p_{-2}(n)$ be defined by

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \frac{1}{f_1^2}. \tag{3.10}$$

It has been shown by Ramanathan [10] that for $n \geq 0$ and $s \in \{2,3,4\}$,

$$p_{-2}(5n+s) \equiv 0 \pmod{5}. \tag{3.11}$$

Using (3.10) in (3.9), we obtain

$$\sum_{n=0}^{\infty} S_2\left(4n+3\right)q^n \equiv 4\frac{f_{10}f_{30}}{f_5 f_{15}}\sum_{n=0}^{\infty} p_{-2}(n)q^{3n} \pmod{20}. \tag{3.12}$$

Congruence (3.8) can be easily obtained by employing (3.11) into (3.12).

**Theorem 3.4.** *For any non–negative integer* $n$*, we have*

$$S_2\left(8n+3\right) \equiv 4f_1 f_2^3 \pmod{8}, \tag{3.13}$$

$$S_2\left(8n+7\right) \equiv 4f_1 f_6^3 \pmod{8}. \tag{3.14}$$

**Proof.** Applying (2.1) into (3.7), we obtain

$$\sum_{n=0}^{\infty} S_2\left(4n+3\right)q^n \equiv 4f_2^3 f_6^2\left(\frac{1}{f_1 f_3}\right) \pmod{8}. \tag{3.15}$$

Substituting (2.6) into (3.15), we obtain

$$\sum_{n=0}^{\infty} S_2\left(4n+3\right)q^n \equiv 4f_2^3 f_6^2\left(\frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2}+q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}\right) \tag{3.16}$$

$$\equiv 4\frac{f_2 f_8^2 f_{12}^5}{f_4 f_6^2 f_{24}^2}+4q\frac{f_4^5 f_{24}^2}{f_2 f_8^2 f_{12}} \pmod{8}. \tag{3.17}$$

Extracting those terms containing the power $q^{2n}$ and $q^{2n+1}$ from both sides of (3.17) and then changing $q$ to $q^{\frac{1}{2}}$, we obtain

$$\sum_{n=0}^{\infty} S_2\left(8n+3\right)q^n \equiv 4\frac{f_1 f_4^2 f_6^5}{f_2 f_3^2 f_{12}^2} \pmod{8} \tag{3.18}$$

and

$$\sum_{n=0}^{\infty} S_2\left(8n+7\right)q^n \equiv 4\frac{f_2^5 f_{12}^2}{f_1 f_4^2 f_6} \pmod{8}. \tag{3.19}$$

Employing (2.1) into (3.18) and (3.19), we arrive at (3.13) and (3.14).

**Theorem 3.5.** *Let $p\geq 5$ be a prime and $\left(\frac{-6}{p}\right)=-1$. Then for all $\alpha\geq 0, n\geq 0$, we have*

$$S_2\left(8p^{2\alpha+2}n+\left(\frac{24i+7p}{3}\right)p^{2\alpha+1}+\frac{2}{3}\right)\equiv 0 \pmod{8}, \tag{3.20}$$

*where $i$ is an integer and $1\leq i\leq p-1$.*

**Proof.** From (3.13), we have

$$\sum_{n=0}^{\infty} S_2\left(8n+3\right)q^n \equiv 4f_1 f_2^3 \pmod{8}. \tag{3.21}$$

Define

$$\sum_{n=0}^{\infty} a(n)q^n = f_1 f_2^3. \tag{3.22}$$

Combining (3.21) and (3.22), we find that

$$S_2\left(8n+3\right)\equiv 4a(n) \pmod{8}. \tag{3.23}$$

Now, we consider the congruence equation

$$\frac{3m^2+m}{2}+2\left(\frac{k^2+k}{2}\right)\equiv\frac{7p^2-7}{24} \pmod{p}, \tag{3.24}$$

which is equivalent to

$$(6m+1)^2 + 6(2k+1)^2 \equiv 0 \pmod{p}, \tag{3.25}$$

where $\frac{-(p-1)}{2} \le m \le \frac{p-1}{2}$, $\frac{-(p-1)}{2} \le k \le \frac{p-3}{2}$ and $p$ is a prime such that $\left(\frac{-6}{p}\right) = -1$. Since $\left(\frac{-6}{p}\right) = -1$, the congruence relation (3.24) holds if and only if $m = \frac{\pm p-1}{6}$ and $k = \frac{p-1}{2}$. Therefore, if we substitute (2.14) and (2.16) into (3.22) and then extract the terms in which the powers of $q$ are $pn + \frac{7p^2-7}{24}$, we arrive at

$$\sum_{n=0}^{\infty} a\left(pn + \frac{7p^2-7}{24}\right) q^{pn+\frac{7p^2-7}{24}} = p(-1)^{\frac{\pm p-1}{6}+\frac{p-1}{2}} q^{\frac{7p^2-7}{24}} f_{p^2} f_{2p^2}^3. \tag{3.26}$$

Dividing $q^{\frac{7p^2-7}{24}}$ on both sides of (3.26) and then replacing $q^p$ by $q$, we find that

$$\sum_{n=0}^{\infty} a\left(pn + \frac{7p^2-7}{24}\right) q^n = p(-1)^{\frac{\pm p-1}{6}+\frac{p-1}{2}} f_p f_{2p}^3, \tag{3.27}$$

which implies that

$$\sum_{n=0}^{\infty} a\left(p^2n + \frac{7p^2-7}{24}\right) q^n = p(-1)^{\frac{\pm p-1}{6}+\frac{p-1}{2}} f_1 f_2^3 \tag{3.28}$$

and for $n \ge 0$,

$$a\left(p^2n + pi + \frac{7p^2-7}{24}\right) = 0, \tag{3.29}$$

where $i$ is an integer and $1 \le i \le p-1$. Combining (3.22) and (3.28), we see that for $n \ge 0$,

$$a\left(p^2n + \frac{7p^2-7}{24}\right) = p(-1)^{\frac{\pm p-1}{6}+\frac{p-1}{2}} a(n). \tag{3.30}$$

By (3.30) and mathematical induction, we deduce that for $n \ge 0$ and $\alpha \ge 0$,

$$a\left(p^{2\alpha}n + \frac{7p^{2\alpha}-7}{24}\right) = p^{\alpha}(-1)^{\alpha\cdot\frac{\pm p-1}{6}+\alpha\cdot\frac{p-1}{2}} a(n). \tag{3.31}$$

Replacing $n$ by $p^2n + pi + \frac{7p^2-7}{24}$ $(i = 1, 2, ..., p-1)$ in (3.31) and using (3.29), we deduce that for $n \ge 0$ and $\alpha \ge 0$,

$$a\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{7p^{2\alpha+2}-7}{24}\right) = 0.$$

Again, replacing $n$ by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{7p^{2\alpha+2}-7}{24}$ $(i = 1, 2, ..., p-1)$ in (3.23), we arrive at (3.20).

**Theorem 3.6.** *Let $p \geq 5$ be a prime and $\left(\frac{-18}{p}\right) = -1$. Then for all $\alpha \geq 0, n \geq 0$, we have*

$$S_2\left(8p^{2\alpha+2} + \left(\frac{24i+19p}{3}\right)p^{2\alpha+1} + \frac{2}{3}\right) \equiv 0 \pmod 8, \tag{3.32}$$

*where $i$ is an integer and $1 \leq i \leq p-1$.*

**Proof.** From (3.14), we have

$$\sum_{n=0}^{\infty} S_2(8n+7)\, q^n \equiv 4f_1f_6^3 \pmod 8. \tag{3.33}$$

Define

$$\sum_{n=0}^{\infty} b(n)q^n = f_1f_6^3. \tag{3.34}$$

Combining (3.33) and (3.34), we see that

$$S_2(8n+7) \equiv 4b(n) \pmod 8. \tag{3.35}$$

Now, we consider the congruence equation

$$\frac{3m^2+m}{2} + 6\left(\frac{k^2+k}{2}\right) \equiv \frac{19p^2-19}{24} \pmod p, \tag{3.36}$$

$$(6m+1)^2 + 18(2k+1)^2 \equiv 0 \pmod p, \tag{3.37}$$

where $\frac{-(p-1)}{2} \leq m \leq \frac{p-1}{2}$, $\frac{-(p-1)}{2} \leq k \leq \frac{p-1}{3}$ and $p$ is a prime such that $(\frac{-18}{p}) = -1$. Since $(\frac{-18}{p}) = -1$, the congruence relation (3.36) holds if and only if $m = \frac{\pm p-1}{6}$ and $k = \frac{p-1}{2}$. Therefore, if we substitute (2.14) and (2.16) into (3.34) and then extract the terms in which the powers of $q$ are $pn + \frac{19p^2-19}{24}$, we arrive at

$$\sum_{n=0}^{\infty} b\left(pn + \frac{19p^2-19}{24}\right) q^{pn+\frac{19p^2-19}{24}} = p(-1)^{\frac{\pm p-1}{6}+\frac{p-1}{2}} q^{\frac{19p^2-19}{24}} f_{p^2}f_{6p^2}^3. \tag{3.38}$$

Dividing $q^{\frac{19p^2-19}{24}}$ on both sides of (3.38) and then replacing $q^p$ by $q$, we find that

$$\sum_{n=0}^{\infty} b\left(pn + \frac{19p^2-19}{24}\right) q^n = p(-1)^{\frac{\pm p-1}{6}+\frac{p-1}{2}} f_pf_{6p}^3, \tag{3.39}$$

which implies that

$$\sum_{n=0}^{\infty} b\left(p^2n+\frac{19p^2-19}{24}\right)q^n = p(-1)^{\frac{\pm p-1}{6}+\frac{p-1}{2}} f_1f_6^3, \tag{3.40}$$

and for $n \geq 0$,

$$b\left(p^2n+pi+\frac{19p^2-19}{24}\right) = 0, \tag{3.41}$$

where $i$ is an integer and $1 \leq i \leq p-1$. Combining (3.34) and (3.40), we see that for $n \geq 0$,

$$b\left(p^2n+\frac{19p^2-19}{24}\right) = p(-1)^{\frac{\pm p-1}{6}+\frac{p-1}{2}} b(n). \tag{3.42}$$

By (3.42) and mathematical induction, we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$b\left(p^{2\alpha}n+\frac{19p^{2\alpha}-19}{24}\right) = p^{\alpha}(-1)^{\alpha\cdot\frac{\pm p-1}{6}+\alpha\cdot\frac{p-1}{2}} b(n). \tag{3.43}$$

Replacing $n$ by $p^2n+pi+\frac{19p^2-19}{24}$ $(i=1,2,\ldots,p-1)$ in (3.43) and using (3.41), we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$b\left(p^{2\alpha+2}n+p^{2\alpha+1}i+\frac{19p^{2\alpha+2}-19}{24}\right) = 0.$$

Again, replacing $n$ by $p^{2\alpha+2}n+p^{2\alpha+1}i+\frac{19p^{2\alpha+2}-19}{24}$ $(i=1,2,\ldots,p-1)$ in (3.35), we arrive at (3.32).

## 4. Three color partition identities of $S(n)$

**Theorem 4.1.** *For any non-negative integer $n$, we have*

$$S_3(3n+1) \equiv 0 \pmod{3}, \tag{4.1}$$

$$S_3(3n+2) \equiv 0 \pmod{3}. \tag{4.2}$$

**Proof.** From (1.5), we have

$$\sum_{n=0}^{\infty} S_3(n)q^n = \frac{f_2^3f_3^3}{f_1^3f_6^6}. \tag{4.3}$$

Substitute (2.11) in (4.3), we obtain

$$\sum_{n=00}^{\infty} S_3(n)q^n = \frac{f_3^2}{f_6^5} + 3q\frac{f_9^5}{f_3^5 f_{18} f_6^2} + 6q^2\frac{f_9^2 f_{18}^2}{f_3^4 f_6^3} + 12q^3\frac{f_{18}^5}{f_6^4 f_3^3 f_9}. \tag{4.4}$$

Extracting those terms containing the power $q^{3n}$, $q^{3n+1}$, $q^{3n+2}$ from both sides of (4.4) and then changing $q$ to $q^{\frac{1}{3}}$, we obtain

$$\sum_{n=00}^{\infty} S_3(3n)q^n = \frac{f_1^2}{f_2^5} + 12q\frac{f_6^5}{f_1^3 f_2^4 f_3}, \tag{4.5}$$

$$\sum_{n=00}^{\infty} S_3(3n+1)q^n = 3\frac{f_3^5}{f_2^2 f_1^5 f_6} \tag{4.6}$$

and

$$\sum_{n=00}^{\infty} S_3(3n+2)q^n = 6\frac{f_3^2 f_6^2}{f_2^3 f_1^4}. \tag{4.7}$$

Congruences (4.1) and (4.2) follow from equations (4.6) and (4.7).

**Theorem 4.2.** *For any non-negative integer $n$, we have*

$$S_3\left(24n\right) \equiv p(n) \pmod 4, \tag{4.8}$$

$$S_3\left(24n+6i\right) \equiv 0 \pmod 4, \tag{4.9}$$

*where $i \in \{1, 2, 3\}$.*

$$S_3\left(48n+3\right) \equiv 2f_1 \pmod 4, \tag{4.10}$$

$$S_3\left(48n+6j+3\right) \equiv 0 \pmod 4, \tag{4.11}$$

*where $j \in \{1, 2, 3, 4, 5, 6, 7\}$.*

$$S_3(12n+6) \equiv 0 \pmod 4. \tag{4.12}$$

**Proof.** From (4.5), we obtain

$$\sum_{n=0}^{\infty} S_3(3n)q^n \equiv \frac{f_1^2}{f_2^5} \pmod 4. \tag{4.13}$$

Substitute (2.4) in (4.13), we obtain

$$\sum_{n=0}^{\infty} S_3(3n)q^n \equiv \frac{f_8^5}{f_2^4 f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_2^4 f_8} \pmod 4. \tag{4.14}$$

Extracting those terms containing the power $q^{2n}$, $q^{2n+1}$ from both sides of (4.14) and then changing $q$ to $q^{\frac{1}{2}}$, we obtain

$$\sum_{n=0}^{\infty} S_3(6n)q^n \equiv \frac{f_4^5}{f_1^4 f_2^2 f_8^2} \pmod 4, \tag{4.15}$$

$$\sum_{n=0}^{\infty} S_3(6n+3)q^n \equiv -2\frac{f_8^2}{f_1^4 f_4} \pmod 4. \tag{4.16}$$

Applying (2.1) into (4.15), we find that

$$\sum_{n=0}^{\infty} S_3(6n)q^n \equiv \frac{1}{f_4} \pmod 4. \tag{4.17}$$

Extracting those terms containing the power $q^{4n}$ from both sides of (4.17) and then changing $q$ to $q^{\frac{1}{4}}$, we arrive at (4.8).

Again, extracting the terms containing the power $q^{4n+i}$, for $i \in \{1, 2, 3\}$ from both side of (4.17), we arrive at (4.9).

Next, apply (2.1) into (4.16), we obtain

$$\sum_{n=0}^{\infty} S_3(6n+3)q^n \equiv 2f_8 \pmod 4. \tag{4.18}$$

Extracting those terms containing the power $q^{8n}$ from both sides of (4.18) and then changing $q$ to $q^{\frac{1}{8}}$, we arrive at (4.10).

Again, extracting the terms containing the power $q^{8n+j}$, for $j \in \{1, 2, 3, 4, 5, 6, 7\}$ from both side of (4.18), we arrive at (4.11).

Substitute (2.5) into (4.15), we obtain

$$\sum_{n=0}^{\infty} S_3(6n)q^n \equiv \frac{f_4^5 f_4^{14}}{f_2^2 f_8^2 f_2^{14} f_8^4} + 4q\frac{f_4^5 f_4^2 f_8^4}{f_2^2 f_8^2 f_2^{10}} \pmod 4. \tag{4.19}$$

Extracting those terms containing the power $q^{2n+1}$ from both sides of (4.19) and then changing $q$ to $q^{\frac{1}{2}}$, we arrive at (4.12).

**Theorem 4.3.** *For any prime $p \equiv 3 \pmod 4, \alpha \geq 0$ and $n \geq 0$, we have*

$$S_3\left(48p^{2\alpha+2} + (48i + 2p)\, p^{2\alpha+1} + 1\right) \equiv 0 \pmod 4, \tag{4.20}$$

*where $i$ is an integer and $1 \leq i \leq p-1$.*

**Proof.** From (4.10), we have

$$\sum_{n=0}^{\infty} S_3\,(48n+3)\, q^n \equiv 2f_1 \pmod 4. \tag{4.21}$$

Define

$$\sum_{n=0}^{\infty} c(n) q^n = f_1. \tag{4.22}$$

Combining (4.21) and (4.22), we see that

$$S_3\,(48n+3) \equiv 2c(n) \pmod 4. \tag{4.23}$$

Now, we consider the congruence equation

$$\frac{3m^2+m}{2} \equiv \frac{p^2-1}{24} \pmod p, \tag{4.24}$$

$$(6m+1)^2 \equiv 0 \pmod p, \tag{4.25}$$

where $\frac{-(p-1)}{2} \leq m \leq \frac{p-1}{2}$ and $p$ is a prime such that $(\frac{-1}{p}) = -1$. Since $(\frac{-1}{p}) = -1$ for $p \equiv 3 \pmod 4$, the congruence relation (4.24) holds if and only if $m = \frac{\pm p-1}{6}$. Therefore, if we substitute (2.14) into (4.22) and then extract the terms in which the powers of $q$ are $pn + \frac{p^2-1}{24}$, we arrive at

$$\sum_{n=0}^{\infty} c\left(pn + \frac{p^2-1}{24}\right) q^{pn+\frac{p^2-1}{24}} = (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}. \tag{4.26}$$

Dividing $q^{\frac{p^2-1}{24}}$ on both sides of (4.26) and then replacing $q^p$ by $q$, we find that

$$\sum_{n=0}^{\infty} c\left(pn + \frac{p^2-1}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} f_p, \tag{4.27}$$

which implies that

$$\sum_{n=0}^{\infty} c\left(p^2n+\frac{p^2-1}{24}\right)q^n=(-1)^{\frac{\pm p-1}{6}}f_1, \tag{4.28}$$

and for $n\geq 0$,

$$c\left(p^2n+pi+\frac{p^2-1}{24}\right)=0, \tag{4.29}$$

where $i$ is an integer and $1\leq i\leq p-1$. Combining (4.22) and (4.28), we see that for $n\geq 0$,

$$c\left(p^2n+\frac{p^2-1}{24}\right)=(-1)^{\frac{\pm p-1}{6}}c(n). \tag{4.30}$$

By (4.30) and mathematical induction, we deduce that for $n\geq 0$ and $\alpha\geq 0$,

$$c\left(p^{2\alpha}n+\frac{p^{2\alpha}-1}{24}\right)=(-1)^{\alpha.\frac{\pm p-1}{6}}c(n). \tag{4.31}$$

Replacing $n$ by $p^2n+pi+\frac{p^2-1}{24}$ $(i=1,2,\ldots,p-1)$ in (4.31) and using (4.30), we deduce that for $n\geq 0$ and $\alpha\geq 0$,

$$c\left(p^{2\alpha+2}n+p^{2\alpha+1}i+\frac{p^{2\alpha+2}-1}{24}\right)=0.$$

Again, replacing $n$ by $p^{2\alpha+2}n+p^{2\alpha+1}i+\frac{p^{2\alpha+2}-1}{24}$ $(i=1,2,\ldots,p-1)$ in (3.35), we arrive at (4.23).

**Theorem 4.4.** *For any non-negative integer $n$, $\alpha$, we have*

$$S_3(9^{\alpha+1}(3n+2)+1)\equiv 0 \pmod 9, \tag{4.32}$$

$$S_3\left(24n+20\right)\equiv 0 \pmod 9. \tag{4.33}$$

**Proof.** From (4.6), we have

$$\sum_{n=0}^{\infty} S_3(3n+1)q^n=3\frac{f_3^5}{f_2^2f_1^5f_6}. \tag{4.34}$$

Applying (2.2) into (4.34), we find that

$$\sum_{n=0}^{\infty} S_3(3n+1)q^n\equiv 3\left(\frac{f_3^2}{f_6}\right)\left(\frac{f_1^2}{f_2}\right)^2 \pmod 9. \tag{4.35}$$

Using $\varphi(-q) = \dfrac{f_1^2}{f_2}$ in (4.35), we get

$$\sum_{n=0}^{\infty} S_3(3n+1)q^n \equiv 3\varphi(-q^3)\varphi^2(-q) \pmod 9. \tag{4.36}$$

Substitute (2.13) into (4.36), we have

$$\sum_{n=0}^{\infty} S_3(3n+1)q^n \equiv 3\varphi(-q^3)\left(\varphi^2\left(-q^9\right) + 4q^2 f^2\left(-q^3, -q^{15}\right)\right. \\ \left. -4q\varphi\left(-q^9\right) f\left(-q^3, -q^{15}\right)\right) \pmod 9. \tag{4.37}$$

Extracting those terms containing the powers $q^{3n}$, $q^{3n+1}$,$q^{3n+2}$ from both sides of (4.37) and then changing $q$ to $q^{\frac{1}{3}}$, we obtain

$$\sum_{n=0}^{\infty} S_3(9n+1)q^n \equiv 3\varphi(-q)\varphi^2\left(-q^3\right) \pmod 9, \tag{4.38}$$

$$\sum_{n=0}^{\infty} S_3(9n+4)q^n \equiv -12\varphi(-q)\varphi\left(-q^3\right) f\left(-q, -q^5\right)) \pmod 9 \tag{4.39}$$

and

$$\sum_{n=0}^{\infty} S_3(9n+7)q^n \equiv 12\varphi(-q) f^2\left(-q, -q^5\right) \pmod 9. \tag{4.40}$$

Substitute (2.13) in (4.38), we obtain

$$\sum_{n=0}^{\infty} S_3(9n+1)q^n \equiv 3\left(\varphi\left(-q^9\right) - 2qf\left(-q^3, -q^{15}\right)\right)\varphi^2\left(-q^3\right)) \pmod 9. \tag{4.41}$$

Extracting those terms containing the powers $q^{3n}$ and $q^{3n+2}$ from both sides of (4.41), we obtain

$$\sum_{n=0}^{\infty} S_3(27n+1)q^n \equiv 3\varphi(-q^3)\varphi^2(-q) \pmod 9 \tag{4.42}$$

and

$$S_3\left(27n+19\right) \equiv 0 \pmod 9. \tag{4.43}$$

From (4.36) and (4.42), we have

$$S_3(27n+1) \equiv S_3(3n+1) \pmod 9. \tag{4.44}$$

By (4.44) and mathematical induction we have, for $n, \alpha \geq 0$,

$$S_3\left(9^{\alpha+1}\cdot 3n+1\right) \equiv S_3(3n+1) \pmod 9. \tag{4.45}$$

Replace $n$ by $9n+6$ for (4.45), we arrive at (4.32).
Extracting those terms containing the power $q^{3n+1}$ from both sides of (4.41) and then changing $q$ to $q^{\frac{1}{3}}$, we obtain

$$\sum_{n=0}^{\infty} S_3(27n+10)q^n \equiv -6\varphi^2(-q)\, f\left(-q,-q^5\right) \pmod 9. \tag{4.46}$$

Extracting those terms containing the power $q^{3n+2}$ from both sides of (4.41) and then changing $q$ to $q^{\frac{1}{3}}$, we arrive at (4.43).

From (4.7), we have

$$\sum_{n=0}^{\infty} S_3(3n+2)q^n = 6\frac{f_3^2 f_6^2}{f_2^3 f_1^4}. \tag{4.47}$$

Applying (2.2) into (4.47), we have

$$\sum_{n=0}^{\infty} S_3(3n+2)q^n \equiv 6f_6\left(\frac{f_3}{f_1}\right) \pmod 9. \tag{4.48}$$

Substitute (2.9) in (4.48), we obtain

$$\sum_{n=0}^{\infty} S_3(3n+2)q^n \equiv 6\left(\frac{f_4 f_6^2 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6^2 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}\right) \pmod 9. \tag{4.49}$$

Extracting those terms containing the power $q^{2n}$ from both sides of (4.49) and then changing $q$ to $q^{\frac{1}{2}}$, we obtain

$$\sum_{n=0}^{\infty} S_3(6n+2)q^n \equiv 6\frac{f_2 f_3^2 f_8 f_{12}^2}{f_1^2 f_4 f_6 f_{24}} \pmod 9. \tag{4.50}$$

Substitute (2.8) in (4.50), we obtain

$$\sum_{n=0}^{\infty} S_3(6n+2)q^n \equiv 6\frac{f_{12}^4 f_4^3}{f_{24}^2 f_2^4} + 12q\frac{f_8^2 f_{12} f_6}{f_2^3} \pmod 9. \tag{4.51}$$

Extracting those terms containing the power $q^{2n}$, $q^{2n+1}$ from both sides of (4.51) and then changing $q$ to $q^{\frac{1}{2}}$ and use (2.2) in (4.51), we obtain

$$\sum_{n=0}^{\infty} S_3(12n+2)q^n \equiv 6\frac{f_6^4 f_2^3}{f_{12}^2 f_1^4} \pmod 9, \tag{4.52}$$

$$\sum_{n=0}^{\infty} S_3(12n+8)q^n \equiv 12\frac{f_4^2 f_2^3 f_3}{f_1^3} \pmod 9. \tag{4.53}$$

Substituting the identity (2.10) in (4.53), we obtain

$$\sum_{n=0}^{\infty} S_3(12n+8)q^n \equiv 12\frac{f_4^8 f_6^3}{f_2^6 f_{12}^2} \pmod 9. \tag{4.54}$$

Extracting those terms containing the power $q^{2n+1}$ from both sides of (4.54), we arrive at (4.33).

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